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Multiple Wilson and Jacobi–Piñeiro polynomials☆

B. Beckermann[a], J. Coussement[b,*,1], W. Van Assche[b]

[a] *Laboratoire de Mathématiques P. Painlevé UMR 8524 (ANO), UFR Mathématiques - M3, Université de Lille 1, 59655 Villeneuve d'Ascq CEDEX, France*

[b] *Department of Mathematics, Katholieke Universiteit Leuven, Celestijnenlaan 200B, B-3001 Leuven, Belgium*

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Abstract

We introduce multiple Wilson polynomials, which give a new example of multiple orthogonal polynomials (Hermite–Padé polynomials) of type II. These polynomials can be written as a Jacobi–Piñeiro transform, which is a generalization of the Jacobi transform for Wilson polynomials, found by Koornwinder. Here we need to introduce Jacobi and Jacobi–Piñeiro polynomials with complex parameters. Some explicit formulas are provided for both Jacobi–Piñeiro and multiple Wilson polynomials, one of them in terms of Kampé de Fériet series. Finally, we look at some limiting relations and construct a part of a multiple AT-Askey table.

Keywords: Multiple orthogonal polynomials; Hypergeometric functions

1. Introduction

Multiple orthogonal polynomials are a generalization of orthogonal polynomials in the sense that they satisfy orthogonality conditions with respect to $r \in \mathbb{N}$ measures $\mu_1, \ldots, \mu_r$.

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* Corresponding author.

E-mail addresses: bbecker@math.univ-lille1.fr (B. Beckermann), jonathan.coussement@wis.kuleuven.ac.be (J. Coussement), Walter.VanAssche@wis.kuleuven.ac.be (W. Van Assche).

[1] J. Coussement is a research assistant of the Fund for Scientific Research–Flanders (Belgium)

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[3,11,22]. In this paper r will always represent the number of weights. Multiple orthogonal polynomials arise naturally in the theory of simultaneous rational approximation, in particular in Hermite–Padé approximation of a system of r (Markov) functions [6,7,20].

There are two types of multiple orthogonal polynomials. In the present paper we only consider multiple orthogonal polynomials of type II. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\vec{n} = (n_1, n_2, \ldots, n_r) \in \mathbb{N}_0^r$ be a vector of r nonnegative integers, which is called a *multi-index* with length $|\vec{n}| := n_1 + n_2 + \cdots + n_r$. Furthermore let $\Gamma_1, \ldots, \Gamma_r$ be the supports of the r measures. A multiple orthogonal polynomial $P_{\vec{n}}$ of type II with respect to the multi-index $\vec{n}$, is a (nontrivial) polynomial of degree $\leqslant |\vec{n}|$ which satisfies the orthogonality conditions

$$\int P_{\vec{n}}(z) z^m \, \mathrm{d}\mu_j(z) = 0, \quad 0 \leqslant m \leqslant n_j - 1, \quad j = 1, \ldots, r. \tag{1.1}$$

Notice that the measures in (1.1) are not necessarily supposed to be positive. In case of a complex orthogonality relation, one usually refers to $P_{\vec{n}}$ as a *formal* multiple orthogonal polynomial.

Eq. (1.1) leads to a system of $|\vec{n}|$ homogeneous linear relations for the $|\vec{n}| + 1$ unknown coefficients of $P_{\vec{n}}$. A basic requirement in the study of such multiple orthogonal polynomials is that there is (up to a scalar multiplicative constant) a unique solution of system (1.1). We call $\vec{n}$ a *normal index* for $\mu_1, \ldots, \mu_r$ if any solution of (1.1) has exactly degree $|\vec{n}|$ (which implies uniqueness). Let $m_k^{(j)} = \int_{\Gamma_j} z^k \mathrm{d}\mu_j(z)$ be the kth moment of the measure μ_j. Further set

$$D_{\vec{n}} = (D_{\vec{n}}^{(1)} \cdots D_{\vec{n}}^{(r)})^T, \tag{1.2}$$

where

$$D_{\vec{n}}^{(j)} = \begin{pmatrix} m_0^{(j)} & m_1^{(j)} & \ldots & m_{n_j-1}^{(j)} \\ m_1^{(j)} & m_2^{(j)} & \ldots & m_{n_j}^{(j)} \\ \vdots & \vdots & & \vdots \\ m_{|\vec{n}|-1}^{(j)} & m_{|\vec{n}|}^{(j)} & \ldots & m_{|\vec{n}|+n_j-2}^{(j)} \end{pmatrix}$$

is an $|\vec{n}| \times n_j$ matrix of moments of the measure μ_j. Then $D_{\vec{n}}$ is the matrix of the linear system (1.1) without the last column. It is known and easily verified that the multi-index $\vec{n} = (n_1, n_2, \ldots, n_r)$ is normal if and only if this matrix has rank $|\vec{n}|$ [22]. When every multi-index is normal we call the system of measures a *perfect system*. For perfect systems, the multiple orthogonal polynomials of type II satisfy a recurrence relation of order $r + 1$. The proof is similar to the proof of the three-terms recurrence relation satisfied by a sequence of orthogonal polynomials, see for instance [3]. Because of this recurrence relation, formal multiple orthogonal polynomials are a useful tool in the spectral theory of non-symmetric linear difference operators [14].

In the literature one can find some examples of multiple orthogonal polynomials with respect to positive measures on the real line which have the same flavor as the classical orthogonal polynomials. Two classes of measures have been analyzed in more detail and are known to form a perfect system, see for instance the monograph [22] or the survey given in [11]. The first class consists of Angelesco systems where the supports of the measures are disjoint intervals. In the second class of so-called AT systems, the supports of the r measures coincide, and the Radon–Nikodym derivatives $\mathrm{d}\mu_j/\mathrm{d}\mu_1$ for $j = 1, \ldots, r$ form an algebraic Chebyshev system [22, Section IV.4] on the convex hull of the support. In the continuous case (where the measures can be written as $\mathrm{d}\mu_j(x) = w_j(x)\mathrm{d}x$, with w_j the weight function of the measure μ_j) there are multiple Hermite, multiple Laguerre I and II, Jacobi–Piñeiro, multiple Bessel, Jacobi–Angeleso, Jacobi–Laguerre and Laguerre–Hermite polynomials, see [4,11,22] and the references therein. Some discrete examples are multiple Charlier, multiple Kravchuk, multiple Meixner I and II and multiple Hahn [5]. All these examples have the same flavor as the classical orthogonal polynomials as there exists a first-order *raising operator*, based on the differential operator D or the difference operators Δ and ∇, and a *Rodrigues formula*. Moreover, there exist differential or difference equations of order $r + 1$ (with polynomial coefficients) [4]. So, they can be called classical. The recurrence relations of order $r+1$ are known explicitly for these examples in the case $r \leqslant 2$. Finally, we mention that there also exist some examples of multiple orthogonal polynomials associated with modified Bessel functions [12,13,26] which can be called classical.

In Section 3.1 we recall the definition of one of these examples, namely Jacobi–Piñeiro polynomials $P_{\vec{n}}^{(\vec{\alpha},\beta)}$, which are orthogonal with respect to the weights $w_j(x) = x^{\alpha_j}(1-x)^{\beta}$ on $[0, 1]$, $\alpha_j, \beta > -1$. These polynomials reduce to the classical Jacobi polynomials (shifted to the interval $[0, 1]$) for $r = 1$. We show in Section 2.1 that Jacobi polynomials remain formal orthogonal polynomials for complex parameters α_1, β, the corresponding complex orthogonality relation being obtained via an analytic extension of the Beta function in both variables. As we show in Section 3.1, also Jacobi–Piñeiro polynomials with complex parameters are formal multiple orthogonal polynomials of type II.

In Section 3.2 we then introduce the formal multiple Wilson polynomials $p_{\vec{n}}(\cdot; a, \vec{b}, c, d)$, which give a new example of formal multiple orthogonal polynomials of type II. They are an extension of the formal Wilson polynomials $p_n(\cdot; a, b, c, d)$ [27,28] for which we recall the definition in Section 2.2. We also mention that, with some conditions on the complex parameters a, b, c, d, we find the Wilson and Racah polynomials on the top of the Askey scheme which have real orthogonality conditions.

The formal multiple Wilson polynomials satisfy complex orthogonality conditions with respect to r Wilson weights

$$\begin{aligned} &w(z; a, b_j, c, d) \\ &\quad = \frac{\Gamma(a+z)\Gamma(a-z)\Gamma(b_j+z)\Gamma(b_j-z)\Gamma(c+z)\Gamma(c-z)\Gamma(d+z)\Gamma(d-z)}{\Gamma(2z)\Gamma(-2z)}, \end{aligned} \tag{1.3}$$

$j = 1, \ldots, r$, where we integrate over the imaginary axis deformed so as to separate the increasing sequences of poles of these weight functions from the decreasing ones. Note that the parameters $a, b_1, \ldots, b_r, c, d$ can take complex values. There are some additional conditions on these complex parameters in order to ensure that the Wilson weights have only simple poles. We prove in Theorem 3.3 that the weight functions (1.3) form a perfect system if $b_i - b_j \notin \mathbb{Z}$ whenever $i \neq j$. In the same theorem we show that, for $\Re(c+d) > 0$ and $0 < |\Re(z)| < \Re(a)$, the formal multiple Wilson polynomials can be written as the

Jacobi–Piñeiro transform

$$p_{\vec{n}}(z^2; a, \vec{b}, c, d) = \kappa_{\vec{n}} \int_0^1 P_{\vec{n}}^{(\vec{\alpha},\beta)}(u) w^{(a-1,\beta)}(u) K(u, z; a, 0, c, d)\, \mathrm{d}u, \tag{1.4}$$

where $\vec{\alpha} = (a + b_1 - 1, \ldots, a + b_r - 1)$ and $\beta = c + d - 1$. Here $\kappa_{\vec{n}}$ is a normalizing constant, $w^{(\alpha,\beta)}(u) = u^\alpha (1-u)^\beta$ the Jacobi weight and

$$K(u, z; a, b, c, d) = \frac{u^{-b-z}}{\Gamma(a-z)\Gamma(a+z)\Gamma(c+d)}\ {}_2F_1\left(\left.\begin{matrix} c-z, d-z \\ c+d \end{matrix}\right| 1-u\right) \tag{1.5}$$

is a kernel function, independent of $\vec{n}$. We use the notation

$${}_pF_q\left(\left.\begin{matrix} \vec{f} \\ \vec{\phi} \end{matrix}\right| z\right) = \sum_{k=0}^{\infty} \frac{\prod_{\ell=1}^{p} (f_\ell)_k}{\prod_{\ell=1}^{q} (\phi_\ell)_k} \frac{z^k}{k!},$$

where $\vec{f} \in \mathbb{C}^p$, $\vec{\phi} \in \mathbb{C}^q$, which is a hypergeometric function. In the scalar case $(r = 1)$ formula (1.4) reduces to a Jacobi transform for the Wilson polynomials which was already found by Koornwinder [18, (3.3)]. We recall this Jacobi transform in Section 2.3 and give a short proof.

The Jacobi–Piñeiro transform (1.4) is the key formula of this paper. In Section 3.1 we obtain two new hypergeometric representations for the Jacobi–Piñeiro polynomials starting from the Rodrigues formula. Applying the Jacobi–Piñeiro transform (1.4) we then also find two explicit formulas for the formal multiple Wilson polynomials (see Section 3.2). One of them is in terms of Kampé de Fériet series [24].

In Section 4 we only consider the cases where we obtain real orthogonality conditions, namely multiple Wilson and multiple Racah. Using appropriate limit relations we then recover hypergeometric representations for the examples of multiple orthogonal polynomials of type II, mentioned above. We also introduce some new examples like multiple dual Hahn, multiple continuous dual Hahn, and multiple Meixner–Pollaczek. As a result, we finally construct a (still incomplete) multiple AT-Askey table which extends the well known Askey scheme for classical orthogonal polynomials to multiple orthogonal polynomials.

2. Jacobi and Wilson polynomials

2.1. Formal Jacobi polynomials

The (shifted) Jacobi polynomials $P_n^{(\alpha,\beta)}$ are a classical example of continuous orthogonal polynomials. Suppose $\alpha, \beta > -1$, then these polynomials are orthogonal with respect to the Jacobi weight function $w^{(\alpha,\beta)}(x) = x^\alpha (1-x)^\beta$ on the interval $[0, 1]$. These polynomials have the explicit expressions [8]

$$P_n^{(\alpha,\beta)}(z) = \frac{(\alpha+1)_n}{n!}\ {}_2F_1\left(\left.\begin{matrix} -n, \alpha+\beta+n+1 \\ \alpha+1 \end{matrix}\right| z\right), \tag{2.1}$$

$$= \frac{(\alpha+1)_n}{n!} (1-z)^{-\beta}\ {}_2F_1\left(\left.\begin{matrix} \alpha+1+n, -\beta-n \\ \alpha+1 \end{matrix}\right| z\right), \tag{2.2}$$

where the second expression is obtained by Euler's formula [1, 15.3.3], [15]. We claim that the (shifted) Jacobi polynomials $P_n^{(\alpha,\beta)}$ are still formal orthogonal polynomials if we allow complex parameters $\alpha, \beta, \alpha+\beta+1 \in \mathbb{C} \setminus \{-1, -2, \ldots\}$ in formula (2.1). This was already mentioned in [19, Theorem 2.1], but we give a different proof.

In order to prove this claim, we require an integral representation for the meromorphic continuation in both variables of the Beta function. Recall, e.g., from [15, §1.1] that the Gamma function Γ has no zeros and is meromorphic in $\mathbb{C}$ with simple poles at $0, -1, -2, \ldots$. Hence the Beta function [15, §1.5]

$$B(z, w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}$$

is meromorphic both in z and w, with simple poles at $z, w = 0, -1, -2, \ldots$. From [15, §1.5] we have the integral representation

$$B(z, w) = \int_0^1 t^{z-1}(1-t)^{w-1}\mathrm{d}t, \quad \Re(z) > 0, \Re(w) > 0. \tag{2.3}$$

In order to obtain a representation valid for general $z, w \in \mathbb{C} \setminus \mathbb{Z}$ (compare with the Pochhammer formula [15, 1.6.(7)]), we follow [16, § 3.4] and consider three sheets $S1$, $S2$ and $S3$ of the appropriate Riemann surface for the function $\zeta^{z-1}(1-\zeta)^{w-1}$ (in the variable ζ) so that

$$\begin{cases} -\pi < \arg(\zeta) < \pi, \ -\pi < \arg(1-\zeta) < \pi, & \text{for } \zeta \in S1 \setminus \{(-\infty, 0] \cup [1, +\infty)\}, \\ 0 < \arg(\zeta) < 2\pi, \ \ \pi < \arg(1-\zeta) < 3\pi, & \text{for } \zeta \in S2 \setminus \{[0, +\infty) \cup [1, +\infty)\}, \\ \pi < \arg(\zeta) < 3\pi, \ \ 0 < \arg(1-\zeta) < 2\pi, & \text{for } \zeta \in S3 \setminus \{(-\infty, 0] \cup (-\infty, 1]\}. \end{cases}$$

Furthermore we choose a closed contour Σ as in Fig. 1 where $\sigma_1, \sigma_2, \sigma_3$ are the transition points between the three sheets. Note that the function $\zeta^{z-1}(1-\zeta)^{w-1}$ is analytic on Σ. For the Beta function we then have

$$B(z, w) = (1 - e^{2\pi i z})^{-1}(1 - e^{2\pi i w})^{-1} \int_\Sigma \zeta^{z-1}(1-\zeta)^{w-1}\,\mathrm{d}\zeta, \quad z, w \in \mathbb{C} \setminus \mathbb{Z}. \tag{2.4}$$

Indeed, if $\Re(z) > 0, \Re(w) > 0$ then the path of integration in (2.4) can be deformed via the usual "shrinking" method in order to approach the interval [0, 1], leading to formula (2.3). In particular, if z and/or w is a strictly positive integer, we can obtain $B(z, w)$ by taking limits in (2.4).

Now we prove our claim that (shifted) Jacobi polynomials with complex parameters are formal orthogonal polynomials.

Theorem 2.1. *Let $\alpha, \beta, \alpha+\beta+1 \in \mathbb{C} \setminus \{-1, -2, \ldots\}$, and consider the (complex) measure $\mu^{(\alpha,\beta)}$ defined by*

$$\int h\,\mathrm{d}\mu^{(\alpha,\beta)} = \lim_{z\to\alpha, w\to\beta} (1 - e^{2\pi i z})^{-1}(1 - e^{2\pi i w})^{-1} \int_\Sigma h(\zeta)\zeta^z(1-\zeta)^w\,\mathrm{d}\zeta.$$

Then $\mu^{(\alpha,\beta)}$ forms a perfect system, with the corresponding nth formal orthogonal polynomial given by the (shifted) Jacobi polynomial $P_n^{(\alpha,\beta)}$.

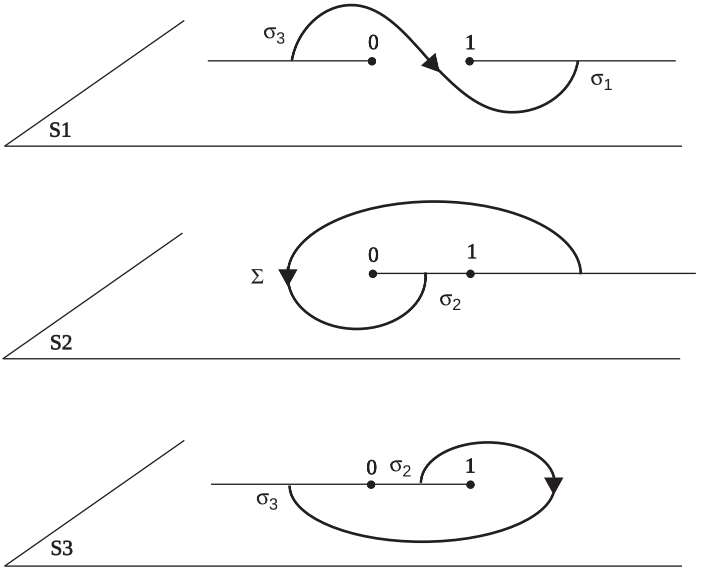

Fig. 1. The contour Σ on the appropriate Riemann surface for $\zeta^{z-1}(1-\zeta)^{w-1}$.

Proof. From (2.4) we obtain for the kth moment

$$\int z^k \, \mathrm{d}\mu^{(\alpha,\beta)}(z) = B(\alpha+1+k, \beta+1).$$

The restrictions on the parameters α and β guarantee that the determinant of the moment matrix,

$$\det(B(\alpha+s+t-1, \beta+1))_{1\leqslant s,t\leqslant n} = \prod_{\ell=1}^{n} \frac{\Gamma(\alpha+\ell)\Gamma(\beta+\ell)}{\Gamma(\alpha+\beta+n+\ell)} \prod_{1\leqslant s<t\leqslant n} (t-s),$$

is different from zero. A proof for this formula uses Theorem 1 and 2 in [9,10] (the moment matrix in the non-shifted case can be found in [25, 6.71.5]). Thus $\mu^{(\alpha,\beta)}$ forms a perfect system. Moreover, for $0\leqslant m\leqslant n-1$ we have, according to (2.1) and (2.4)

$$\begin{aligned}
&\int (1-z)^m P_n^{(\alpha,\beta)}(z) \, \mathrm{d}\mu^{(\alpha,\beta)}(z) \\
&\quad = \frac{(\alpha+1)_n}{n!} \sum_{k=0}^{n} \frac{(-n)_k(\alpha+\beta+n+1)_k}{k!(\alpha+1)_k} B(\alpha+k+1, \beta+m+1) \\
&\quad = \frac{(-1)^n\Gamma(\alpha+n+1)\Gamma(\beta+m+1)}{\Gamma(\alpha+\beta+n+1)} \sum_{k=0}^{n} \frac{(-1)^{n-k}}{k!(n-k)!} \prod_{\ell=m+2}^{n} (\alpha+\beta+k+\ell).
\end{aligned}$$

The sum on the right-hand side is the divided difference $g_{m,n}[0, 1, \ldots, n]$ of the polynomial $g_{m,n}(z) = \prod_{\ell=m+2}^{n}(\alpha+\beta+z+\ell)$ of degree $n-m-1$. Since $n-m-1 < n$, this is equal to 0, which proves that the Jacobi polynomial is a formal orthogonal polynomial with respect to $\mu^{(\alpha,\beta)}$. $\square$

2.2. Formal Wilson polynomials

In [27] Wilson introduced the (formal) Wilson polynomials

$$\begin{aligned} &p_n(z^2; a, b, c, d) \\ &\quad = (a+b)_n (a+c)_n (a+d)_n \\ &\qquad \times {}_4F_3\left(\left. \begin{matrix} -n, a+b+c+d+n-1, a-z, a+z \\ a+b, a+c, a+d \end{matrix} \right| 1 \right). \end{aligned} \tag{2.5}$$

By using Whipple's identities [2, Theorem 3.3.3], one can show that these formal Wilson polynomials are symmetric in all four complex parameters a, b, c, d. Furthermore, with some conditions on these four parameters, the polynomials satisfy a complex orthogonality with respect to the Wilson weight function

$$\begin{aligned} &w(z; a, b, c, d) \\ &\quad = \frac{\Gamma(a+z)\Gamma(a-z)\Gamma(b+z)\Gamma(b-z)\Gamma(c+z)\Gamma(c-z)\Gamma(d+z)\Gamma(d-z)}{\Gamma(2z)\Gamma(-2z)}. \end{aligned}$$

Suppose that

$$2a, a+b, a+c, a+d, 2b, b+c, b+d, 2c, c+d, 2d \notin \{0, -1, -2, \ldots\}, \tag{2.6}$$

so that the Wilson weight has only simple poles, and that

$$a+b+c+d \notin \{0, -1, -2, \ldots\}. \tag{2.7}$$

Furthermore let $\mathcal{C}$ denote the contour obtained by deforming the imaginary axis so as to separate the increasing sequences of poles ($\{a+k\}_{k=0}^{\infty}, \{b+k\}_{k=0}^{\infty}, \{c+k\}_{k=0}^{\infty}, \{d+k\}_{k=0}^{\infty}$) from the decreasing ones ($\{-a-k\}_{k=0}^{\infty}, \{-b-k\}_{k=0}^{\infty}, \{-c-k\}_{k=0}^{\infty}, \{-d-k\}_{k=0}^{\infty}$), and define the Wilson measure $\mu^{(a,b,c,d)}$ by

$$\int h \, \mathrm{d}\mu^{(a,b,c,d)} = \int_{\mathcal{C}} h(z^2) w(z; a, b, c, d) \, \mathrm{d}z. \tag{2.8}$$

Wilson [27] proves the complex orthogonality relations

$$\int p_m(z; a, b, c, d) p_n(z; a, b, c, d) \mathrm{d}\mu^{(a,b,c,d)}(z) = \delta_{m,n} 2i M_n,$$

where

$$\begin{aligned} M_n = {} & 2\pi n! \, (a+b+c+d+n-1)_n \, \Gamma(a+b+n) \\ & \times \frac{\Gamma(a+c+n)\Gamma(a+d+n)\Gamma(b+c+n)\Gamma(b+d+n)\Gamma(c+d+n)}{\Gamma(a+b+c+d+2n)}. \end{aligned}$$

The singleton system $\mu^{(a,b,c,d)}$ is perfect; this will follow from Lemma 3.4 below.

In some cases we obtain real orthogonality conditions with respect to positive measures on the real line [27]. If $\Re(a), \Re(b), \Re(c), \Re(d) > 0$ and a, b, c and d are real except possibly for conjugate pairs, then $\mathcal{C}$ can be taken to be the imaginary axis and we obtain the real orthogonality

$$\int_0^\infty p_m(-x^2; a, b, c, d) p_n(-x^2; a, b, c, d) \times \left| \frac{\Gamma(a+ix)\Gamma(b+ix)\Gamma(c+ix)\Gamma(d+ix)}{\Gamma(2ix)} \right|^2 \mathrm{d}x = \delta_{m,n} M_n.$$

Another case is when $a < 0$ and $a+b, a+c, a+d$ are positive or a pair of complex conjugates occurs with positive real parts (where the condition $2a \notin \{0, -1, -2, \ldots\}$, following from (2.6), is removable). We then get the same positive continuous weight function where some positive point masses are added. In these cases we obtain the Wilson polynomials $W_n(z^2; a, b, c, d) := p_n(-z^2; a, b, c, d)$ (see, e.g., [17]), which are real when z^2 is real.

Finally in the case that one of $a+b, a+c, a+d$ is equal to $-N+\varepsilon$, with N a nonnegative integer, one obtains a purely discrete orthogonality after dividing by $\Gamma(-N+\varepsilon)$ and letting $\varepsilon \to 0$. Taking the substitution $z \to z+a$ and the change of variables $\alpha = a+b-1$, $\beta = c+d-1$, $\gamma = a+d-1$, $\delta = a-d$ we then find the Racah polynomials up to a multiplicative constant:

$$R_n(\lambda(z); \alpha, \beta, \gamma, \delta) = {}_4F_3\left(\left. \begin{matrix} -n, n+\alpha+\beta+1, -z, z+\gamma+\delta+1 \\ \alpha+1, \beta+\delta+1, \gamma+1 \end{matrix} \right| 1 \right),$$

where $\lambda(z) = z(z+\gamma+\delta+1)$ and $\alpha+1 = -N$ or $\beta+\delta+1 = -N$ or $\gamma+1 = -N$. (Here we assume a translation of the conditions (2.6) and (2.7)). The Racah polynomials satisfy the discrete orthogonality

$$\sum_{k=0}^{N} \frac{(\alpha+1)_k(\gamma+1)_k(\beta+\delta+1)_k(\gamma+\delta+1)_k((\gamma+\delta+3)/2)_k}{(-\alpha+\gamma+\delta+1)_k(-\beta+\gamma+1)_k((\gamma+\delta+1)/2)_k(\delta+1)_k k!} \times R_n(\lambda(k); \alpha, \beta, \gamma, \delta) R_m(\lambda(k); \alpha, \beta, \gamma, \delta) = 0,$$

$m \neq n$. Necessary and sufficient conditions for the positivity of the weights are quite messy. An example of sufficient conditions is given in [27, (3.5)], namely

$$\beta+\delta+1 = -N, \quad \gamma+\delta+1 > -1, \quad \alpha > -1, \quad \gamma+\delta+1 > -\alpha$$

and either

$$\gamma+1 > -N \quad \text{or} \quad \delta+1 > -N.$$

The formal Wilson polynomials contain as limiting cases several families of orthogonal polynomials like Hahn, dual Hahn, Meixner, Krawtchouk, Charlier, continuous Hahn, continuous dual Hahn, Meixner–Pollaczek, Jacobi, Laguerre and Hermite polynomials [27, Section 4]; [17, Chapter 2]. Together they form the Askey scheme of hypergeometric orthogonal polynomials. Moreover, there exist q-analogues [17] for all these polynomials.

2.3. Formal Wilson polynomials as a Jacobi transform

We now recall an integral relation between the Jacobi and the formal Wilson polynomials. We also give a short proof which will help us to find explicit expressions for the formal multiple Wilson polynomials in the next section.

Theorem 2.2 (*Koornwinder*). *Suppose that conditions* (2.6) *and* (2.7) *hold, and that* $\Re(c+d) > 0$, $0 < |\Re(z)| < \Re(a)$. *Then we have*

$$p_n(z^2; a, b, c, d) = \kappa_n \int_0^1 P_n^{(\alpha,\beta)}(u) w^{(\alpha,\beta)}(u) K(u, z; a, b, c, d)\, \mathrm{d}u, \tag{2.9}$$

with $\alpha = a + b - 1$ *and* $\beta = c + d - 1$, *the Jacobi weight function* $w^{(\alpha,\beta)}(u) = u^\alpha(1-u)^\beta$, *the constant* $\kappa_n = n!\, \Gamma(a + c + n)\Gamma(a + d + n)$ *and the kernel* (1.5).

Remark 2.3. Koornwinder mentioned this Jacobi transform for the Wilson polynomials in [18, (3.3)]. One can show that our representation coincides with that of Koornwinder by making suitable parameter changes and the substitution $1 - \tanh^2 s \to u$. Notice also that (2.9) is a particular case of a formula due to Meijer [21, p.103].

Proof. By comparing the explicit formulas (2.1) and (2.5) we see that it is sufficient to prove that, for $\ell \in \mathbb{N}$,

$$\int_0^1 u^\ell w^{(a-1,c+d-1)}(u) K(u, z; a, 0, c, d)\, \mathrm{d}u = \frac{(a-z)_\ell (a+z)_\ell}{\Gamma(a+c+\ell)\Gamma(a+d+\ell)}. \tag{2.10}$$

By definition of the kernel (1.5) we have

$$K(u, z; a, 0, c, d) = (a-z)_\ell (a+z)_\ell\, K(u, z; a+\ell, 0, c, d),$$

and $u^\ell w^{(a-1,c+d-1)}(u) = w^{(a+\ell-1,c+d-1)}(u)$. So, by replacing $a+\ell$ by a, we see that it remains to show that, for $0 < |\Re(z)| < \Re(a)$ and $\Re(c+d) > 0$,

$$\int_0^1 w^{(a-1,c+d-1)}(u) K(u, z; a, 0, c, d)\, \mathrm{d}u = \frac{1}{\Gamma(a+c)\Gamma(a+d)}. \tag{2.11}$$

Euler's formula gives the symmetry $K(u, -z; a+\ell, 0, c, d) = K(u, z; a+\ell, 0, c, d)$. So, it is enough to prove (2.11) for $0 < \Re(z) < \Re(a)$ and $\Re(c+d) > 0$.

Denoting the left-hand side of (2.11) by Z, we have by definition of the kernel

$$Z = \frac{1}{\Gamma(a-z)\Gamma(a+z)\Gamma(c+d)} \times \int_0^1 w^{(a-z-1,c+d-1)}(u)\ {}_2F_1\left(\left.\begin{matrix} c-z, d-z \\ c+d \end{matrix}\right| 1-u\right) \mathrm{d}u.$$

In order to change the order of summation and integration, we notice that, if $\Re(C-A-B) > 0$,

$$\lim_{y\uparrow 1} \max_{v\in[0,1]} \left| {}_2F_1\left(\left.\begin{matrix} A, B \\ C \end{matrix}\right| yv\right) - {}_2F_1\left(\left.\begin{matrix} A, B \\ C \end{matrix}\right| v\right)\right| = 0,$$

which follows by observing that

$$\frac{\Gamma(A+k)\Gamma(B+k)}{\Gamma(C+k)\Gamma(1+k)} = k^{A+B-C-1}(1+\mathcal{O}(k^{-1})), \quad k\to\infty,$$

by Stirling's formula. Consequently, using the assumptions $\Re(2z) > 0$, $\Re(a-z) > 0$, $\Re(c+d) > 0$ together with (2.3) we obtain by uniform convergence

$$\begin{aligned} Z &= \frac{1}{\Gamma(a-z)\Gamma(a+z)\Gamma(c+d)} \\ &\quad \times \lim_{y\uparrow 1} \int_0^1 w^{(a-z-1,c+d-1)}(u)\, {}_2F_1\left(\left. \begin{matrix} c-z, d-z \\ c+d \end{matrix} \right| y(1-u)\right) \mathrm{d}u \\ &= \lim_{y\uparrow 1} \sum_{k=0}^{\infty} \frac{(c-z)_k (d-z)_k\, y^k}{\Gamma(a-z)\Gamma(a+z)\Gamma(c+d+k)k!} \int_0^1 w^{(a-z-1,c+d+k-1)}(u)\,\mathrm{d}u \\ &= \lim_{y\uparrow 1} \frac{1}{\Gamma(a+z)\Gamma(a+c+d-z)}\, {}_2F_1\left(\left. \begin{matrix} c-z, d-z \\ a+c+d-z \end{matrix} \right| y\right). \end{aligned}$$

The assumption on the parameters ensures that $\Re(a+c+d-z-(c-z+d-z)) > 0$, and hence we get from the Gauss formula [2, Theorem 2.2.2]; [1, 15.1.20]

$$Z = \frac{1}{\Gamma(a+z)\Gamma(a+c+d-z)}\, {}_2F_1\left(\left. \begin{matrix} c-z, d-z \\ a+c+d-z \end{matrix} \right| 1\right) = \frac{1}{\Gamma(a+d)\Gamma(a+c)},$$

as claimed in (2.11). □

3. Formal multiple Wilson as a Jacobi–Piñeiro transform

3.1. Jacobi–Piñeiro with complex parameters

The Jacobi–Piñeiro polynomials are defined by a Rodrigues formula

$$P_{\vec{n}}^{(\vec{\alpha},\beta)}(z) = \frac{1}{\vec{n}!}(1-z)^{-\beta} \prod_{j=1}^{r} \left(z^{-\alpha_j} \frac{\mathrm{d}^{n_j}}{\mathrm{d}z^{n_j}} z^{n_j+\alpha_j} \right) (1-z)^{|\vec{n}|+\beta}, \tag{3.1}$$

where $\vec{n}! = \prod_{j=1}^{r} n_j!$. It is well known [23] that, provided that $\alpha_j > -1$, $\beta > -1$ and $\alpha_i - \alpha_j \notin \mathbb{Z}$ for $i \neq j$, the Jacobi–Piñeiro polynomials are multiple orthogonal polynomials of type II with respect to the (positive) Jacobi weights $w^{(\alpha_j,\beta)}$, $j = 1,\ldots,r$, on the interval $[0, 1]$. Notice that these weights form an AT system, see e.g. [22], and hence we obtain a perfect system of measures. Similar to Theorem 2.1, we show that for complex parameters we keep formal multiple orthogonal polynomials of type II. Here we use the measures $\mu^{(\alpha_j,\beta)}$ of Theorem 2.1 which have as support the contour Σ, but can be reduced to complex Jacobi weights $w^{(\alpha_j,\beta)}$ on the interval $[0, 1]$ in the case $\Re(\alpha_j) > -1$, $\Re(\beta) > -1$.

Theorem 3.1. *Let $\alpha_j, \beta, \alpha_j+\beta \in \mathbb{C}\setminus\{-1,-2,\ldots\}$ and $\alpha_i - \alpha_j \notin \mathbb{Z}$ for $i \neq j$. Then the measures $\mu^{(\alpha_j,\beta)}$, $j = 1,\ldots,r$, defined as in Theorem 2.1, form a perfect system.*

The corresponding formal multiple orthogonal polynomial of type II *with respect to the multi-index $\vec{n}$ is given by* (3.1).

Proof. From Theorem 1 and 2 in [9,10] we obtain for the determinant of the matrix of moments (1.2) the expression

$$D_{\vec{n}}^{\vec{\alpha},\beta} = \left(\prod_{\ell=1}^{|\vec{n}|} \Gamma(\beta+\ell)\right)\left(\prod_{j=1}^{r}\prod_{\ell=1}^{n_j} \frac{\Gamma(\alpha_j+\ell)}{\Gamma(\alpha_j+\beta+\vec{n}+\ell)}\right) \times \left(\prod_{j=1}^{r}\prod_{1\leqslant s<t\leqslant n_j}(t-s)\right)\left(\prod_{1\leqslant i<j\leqslant r}\prod_{s=1}^{n_i}\prod_{t=1}^{n_j}(\alpha_j-\alpha_i+t-s)\right).$$

For our choice of parameters, this expression is different from zero, and hence every multi-index is normal.

Our claim on the (formal) orthogonality of Jacobi–Piñeiro polynomials will be shown by induction on r. For $r=1$, Eq. (3.1) reduces to (2.1), and the claim follows from Theorem 2.1. For $r>1$, we observe first that, by the Rodrigues formula (3.1),

$$P_{\vec{n}}^{(\vec{\alpha},\beta)}(z) = \frac{z^{-\alpha_1}(1-z)^{-\beta}}{n_1!}\frac{\mathrm{d}^{n_1}}{\mathrm{d}z^{n_1}}\left(z^{\alpha_1+n_1}(1-z)^{\beta+n_1}P_{\vec{n}^{(1)}}^{(\vec{\alpha}^{(1)},\beta+n_1)}(z)\right), \tag{3.2}$$

where we denote by $\vec{v}^{\,(j)}$ the vector $\vec{v}$ without the jth component. By the induction hypothesis, $P_{\vec{n}^{(1)}}^{(\vec{\alpha}^{(1)},\beta+n_1)}$ is a polynomial of degree $|\vec{n}|-n_1$, and thus there exist scalars c_ℓ with

$$P_{\vec{n}^{(1)}}^{(\vec{\alpha}^{(1)},\beta+n_1)}(z) = \sum_{\ell=0}^{|\vec{n}|-n_1} c_\ell\, P_\ell^{(\alpha_1+n_1,\beta+n_1)}(z).$$

From the Rodrigues formula for $r=1$ and (3.2) we then conclude

$$P_{\vec{n}}^{(\vec{\alpha},\beta)}(z) = \sum_{\ell=0}^{|\vec{n}|-n_1}\binom{n_1+\ell}{\ell}c_\ell\, P_{\ell+n_1}^{(\alpha_1,\beta)}(z),$$

which implies that $P_{\vec{n}}^{(\vec{\alpha},\beta)}$ is a polynomial of degree $|\vec{n}|$ for which

$$\int z^m P_{\vec{n}}^{(\vec{\alpha},\beta)}(z)\,\mathrm{d}\mu^{(\alpha_1,\beta)}(z) = 0, \quad 0\leqslant m\leqslant n_1-1,$$

by Theorem 2.1. The other orthogonality conditions are obtained by observing that (3.1) remains invariant if one changes the order in the product. □

With help of the Leibniz rule applied to the Rodrigues formula, the authors in [11] derive for $r=2$ an explicit expression in terms of 2 sums. We now give a generalization of this formula for $r \geqslant 2$, using the notation

$$\mathcal{M}^{p;m}_{q,\vec{n}}\left(\begin{array}{c} \vec{f}; \vec{g}_1 : \cdots : \vec{g}_m \\ \vec{\phi}; \vec{\psi}_1 : \cdots : \vec{\psi}_m \end{array}\middle| \vec{z}\right) := \underbrace{\sum_{k_1=0}^{n_1} \cdots \sum_{k_r=0}^{n_r}}_{r \text{ sums}} \frac{\prod_{\ell=1}^{p} (f_\ell)_{|\vec{k}|}}{\prod_{\ell=1}^{q} (\phi_\ell)_{|\vec{k}|}}$$

$$\times \frac{\prod_{i=1}^{m} (g_{i,1})_{|\vec{k}|-k_1} (g_{i,2})_{|\vec{k}|-k_1-k_2} \cdots (g_{i,r-1})_{k_r}}{\prod_{i=1}^{m} (\psi_{i,1})_{|\vec{k}|-k_1} (\psi_{i,2})_{|\vec{k}|-k_1-k_2} \cdots (\psi_{i,r-1})_{k_r}} \prod_{j=1}^{r} (-n_j)_{k_j} \frac{z_j^{k_j}}{k_j!}, \tag{3.3}$$

where $\vec{k} = (k_1, \ldots, k_r)$, $\vec{n} \in \mathbb{N}_0^r = (\mathbb{N} \cup \{0\})^r$, $\vec{g}_1, \ldots, \vec{g}_m, \vec{\psi}_1, \ldots, \vec{\psi}_m \in \mathbb{C}^{r-1}$ and $\vec{f} \in \mathbb{C}^p$, $\vec{\phi} \in \mathbb{C}^q$. We also give in (3.5) another new explicit expression for the formal Jacobi–Piñeiro polynomials which reduces to (2.2) if $r = 1$.

Theorem 3.2. *Let $\vec{e} = (1, \ldots, 1)$ be a multi-index of length r and $s(\vec{n}) = (n_1, n_1 + n_2, \ldots, |\vec{n}|)$. Denote by $\vec{v}^{\,(j)}$ the vector $\vec{v}$ without the jth component. For the Jacobi–Piñeiro polynomials we have the hypergeometric representations*

$$\begin{aligned} P_{\vec{n}}^{(\vec{\alpha},\beta)}(z) &= \frac{(\vec{\alpha}+\vec{e})_{\vec{n}}}{\vec{n}!} \\ &\quad \times \mathcal{M}^{1;2}_{1,\vec{n}}\left(\begin{array}{c} (\alpha_1+\beta+n_1+1); (\vec{\alpha}+\vec{n}+\vec{e})^{(r)} : (\vec{\alpha}+s(\vec{n})+(\beta+1)\vec{e})^{(1)} \\ (\alpha_1+1); (\vec{\alpha}+\vec{e})^{(1)} : (\vec{\alpha}+s(\vec{n})+(\beta+1)\vec{e})^{(r)} \end{array}\middle| z\,\vec{e}\right) \end{aligned} \tag{3.4}$$

and

$$P_{\vec{n}}^{(\vec{\alpha},\beta)}(z) = \frac{(\vec{\alpha}+\vec{e})_{\vec{n}}}{\vec{n}!} (1-z)^{-\beta} {}_{r+1}F_r\left(\begin{array}{c} \vec{\alpha}+\vec{n}+\vec{e}, -\beta-|\vec{n}| \\ \vec{\alpha}+\vec{e} \end{array}\middle| z\right), \tag{3.5}$$

where $\vec{n}! = \prod_{j=1}^{r} n_j!$ and $(\vec{\alpha}+\vec{e})_{\vec{n}} = \prod_{j=1}^{r} (\alpha_j+1)_{n_j}$.

Proof. We prove (3.4) and (3.5) by induction on r. For $r = 1$, Eqs. (3.4) and (3.5) reduce to (2.1) and (2.2), respectively.

In the case $r \geqslant 2$, we use formula (3.2), where the induction hypothesis enables us to express the right-hand polynomial $P_{\vec{n}^{(1)}}^{(\vec{\alpha}^{(1)},\beta+n_1)}$ as a hypergeometric sum. After exchanging the order of summation and differentiation (which is only possible for $|z| < 1$ in case of formula (3.5)), we apply the formulas

$$\begin{aligned} &z^{-\alpha_1}(1-z)^{-\beta} \frac{\mathrm{d}^{n_1}}{\mathrm{d}z^{n_1}} \left(z^{\alpha_1+n_1+|\vec{k}^{(1)}|}(1-z)^{\beta+n_1}\right) \\ &\quad = (\alpha_1+1)_{n_1} \sum_{k_1=0}^{n_1} \frac{(\alpha_1+\beta+n_1+1)_{|\vec{k}|}}{(\alpha_1+1)_{|\vec{k}|}} \frac{(\alpha_1+n_1+1)_{|\vec{k}|-k_1}}{(\alpha_1+\beta+n_1+1)_{|\vec{k}|-k_1}} \frac{(-n_1)_{k_1} z^{|\vec{k}|}}{k_1!} \end{aligned} \tag{3.6}$$

and

$$z^{-\alpha_1}\frac{\mathrm{d}^{n_1}}{\mathrm{d}z^{n_1}}z^{\alpha_1+n_1+k}=\frac{(\alpha_1+1)_{n_1}(\alpha_1+n_1+1)_k}{(\alpha_1+1)_k}\,z^k \tag{3.7}$$

to obtain the right-hand expressions of (3.4), and (3.5), respectively. It remains to prove claims (3.6) and (3.7), the second one being obvious. We observe that the left-hand side of (3.6) can be transformed using the Rodrigues formula for $r=1$ and (2.1), leading to the expression

$$\begin{aligned}
&z^{|\vec{k}^{(1)}|}(\alpha_1+|\vec{k}^{(1)}|+1)_{n_1}\,{}_2F_1\left(\left.\begin{matrix}-n_1,\,\alpha_1+|\vec{k}^{(1)}|+\beta+n_1+1\\ \alpha_1+|\vec{k}^{(1)}|+1\end{matrix}\right|z\right)\\
&\quad=(\alpha_1+|\vec{k}^{(1)}|+1)_{n_1}\sum_{k_1=0}^{n_1}\frac{(-n_1)_{k_1}(\alpha_1+|\vec{k}^{(1)}|+\beta+n_1+1)_{k_1}}{(\alpha_1+|\vec{k}^{(1)}|+1)_{k_1}}\frac{z^{|\vec{k}|}}{k_1!}\\
&\quad=(\alpha_1+1)_{n_1}\sum_{k_1=0}^{n_1}\frac{(\alpha_1+\beta+n_1+1)_{|\vec{k}|}}{(\alpha_1+1)_{|\vec{k}|}}\frac{(\alpha_1+n_1+1)_{|\vec{k}|-k_1}}{(\alpha_1+\beta+n_1+1)_{|\vec{k}|-k_1}}\frac{(-n_1)_{k_1}z^{|\vec{k}|}}{k_1!},
\end{aligned}$$

as claimed in (3.6). □

In the above proof we have shown implicitly that the right hand-side of (3.5) is a polynomial of degree $\leqslant|\vec{n}|$ in z.

Multiple orthogonal polynomials satisfy a recurrence relation of order $r+1$, see, e.g., [20, §24]; [3]. With the explicit formula (3.4) it is possible to compute the recurrence coefficients by comparing the highest coefficients in the recurrence relation, see [4,11] for $r=2$.

3.2. *Formal multiple Wilson polynomials*

We now consider r Wilson weights

$$w(\cdot;a,b_j,c,d),\qquad j=1,\ldots,r,\qquad b_i-b_j\notin\mathbb{Z},\qquad i\neq j, \tag{3.8}$$

defined as in (1.3), that is, we change only one parameter (recall the symmetry of the Wilson weights in all four parameters). As in the scalar case, which is the family of Wilson polynomials, we suppose that for $j=1,\ldots,r$

$$2a,a+b_j,a+c,a+d,2b_j,b_j+c,b_j+d,2c,c+d,2d\notin\{0,-1,-2,\ldots\}, \tag{3.9}$$

so that the r Wilson weights have only simple poles, and that

$$a+b_j+c+d\notin\{0,-1,-2,\ldots\}. \tag{3.10}$$

As in the scalar case, we write $\mu^{(a,b_j,c,d)}$ for the resulting measures, where it is possible to choose a joint contour $\mathcal{C}$ which is the imaginary axis deformed so as to separate the increasing sequences of poles $(\{a+k\}_{k=0}^{\infty},\{b_1+k\}_{k=0}^{\infty},\ldots,\{b_r+k\}_{k=0}^{\infty},\{c+k\}_{k=0}^{\infty},\{d+k\}_{k=0}^{\infty})$ from the decreasing ones $(\{-a-k\}_{k=0}^{\infty},\{-b_1-k\}_{k=0}^{\infty},\ldots,\{-b_r-k\}_{k=0}^{\infty},\{-c-k\}_{k=0}^{\infty},\{-d-k\}_{k=0}^{\infty})$.

Let us show that the (possibly complex) Wilson measures $\mu^{(a,b_j,c,d)}$ form a perfect system. The corresponding multiple orthogonal polynomials will then be referred to as formal multiple Wilson polynomials. A basic observation in what follows is that, under some additional conditions, the formal multiple Wilson polynomials can be written as a Jacobi–Piñeiro transform, similar to (2.9).

Theorem 3.3. *Suppose that* (3.9) *and* (3.10) *hold and that* $b_i - b_j \notin \mathbb{Z}, i \neq j$. *The Wilson measures* $\mu^{(a,b_j,c,d)}$, $j = 1, \ldots, r$, *then form a perfect system. Furthermore, if* $\Re(a) > 0$ *and* $\Re(c+d) > 0$, *the formal multiple Wilson polynomial with multi-index* $\vec{n}$ *can be written as*

$$p_{\vec{n}}(z^2; a, \vec{b}, c, d) = \kappa_{\vec{n}} \int_0^1 P_{\vec{n}}^{(\vec{\alpha},\beta)}(u) w^{(a-1,\beta)}(u) K(u, z; a, 0, c, d)\, \mathrm{d}u, \tag{3.11}$$

for $0 < |\Re(z)| < \Re(a)$, *where* $\vec{\alpha} = (a + b_1 - 1, \ldots, a + b_r - 1) = (a-1)\vec{e} + \vec{b}$, *with* $\vec{e} = (1, \ldots, 1) \in \mathbb{R}^r$, *and* $\beta = c + d - 1$. *The normalizing constant* $\kappa_{\vec{n}} = \vec{n}!\, \Gamma(a + c + |\vec{n}|)\Gamma(a + d + |\vec{n}|)$ *is chosen so that it corresponds with* (2.9) *in the case* $r = 1$ *and the kernel* $K(u, z; a, b, c, d)$ *is defined as in* (1.5).

Before we prove this theorem we need some technical lemmas.

Lemma 3.4. *A system of r measures* $\mu_1, \ldots, \mu_r$ *is perfect if and only if, for every multi-index* $\vec{n}$, *there exists a polynomial* $P_{\vec{n}}$ *of exactly degree* $|\vec{n}|$ *such that*

$$\int P_{\vec{n}}(z) z^m \, \mathrm{d}\mu_j(z) = 0, \quad 0 \leqslant m \leqslant n_j - 1, \quad j = 1, \ldots, r, \tag{3.12}$$

and

$$\int P_{\vec{n}}(z) z^{n_j} \, \mathrm{d}\mu_j(z) \neq 0, \quad j = 1, \ldots, r. \tag{3.13}$$

In this case, $P_{\vec{n}}$ *is the* (*up to normalization unique*) *multiple orthogonal polynomial of type* II *with respect to* $\vec{n}$.

Proof. Suppose first that $\mu_1, \ldots, \mu_r$ is perfect, and take as $P_{\vec{n}}$ the multiple orthogonal polynomial of type II with respect to $\vec{n}$. Then it only remains to verify (3.13). Suppose the contrary, that is, $\int P_{\vec{n}}(z) z^{n_j} \mathrm{d}\mu_j(z) = 0$ for some j. Then $P_{\vec{n}}$ is also a multiple orthogonal polynomial of type II with respect to $\vec{n} + \vec{e}_j$, in contrast to the normality of the multi-index $\vec{n} + \vec{e}_j$. Here $\vec{e}_j$ is the jth unit vector in $\mathbb{R}^r$.

We will prove the other implication of this lemma by showing that every $\vec{n}$ is normal by induction on the length $|\vec{n}|$. The multi-index $\vec{0}$ is always normal, suppose therefore that $\vec{n}$ is of length $\geqslant 1$, with its jth component strictly greater than 0. Let $R_{\vec{n}}$ be a multiple orthogonal polynomial for $\vec{n}$. If $R_{\vec{n}}$ would have degree strictly less than $|\vec{n}|$, then it would be a multiple orthogonal polynomial for the multi-index $\vec{n} - \vec{e}_j$ with length $|\vec{n}| - 1$. By normality of $\vec{n} - \vec{e}_j$ we then have that there exists a nonzero constant c so that $R_{\vec{n}} = c P_{\vec{n} - \vec{e}_j}$. Thus $\int R_{\vec{n}}(z) z^{n_j - 1} \mathrm{d}\mu_j(z) \neq 0$ by (3.13), in contradiction to the orthogonality relation (3.12) for $R_{\vec{n}}$. As a consequence, $R_{\vec{n}}$ has the precise degree $|\vec{n}|$, and thus $\vec{n}$ is normal. $\square$

Lemma 3.5. *Suppose that the singleton systems μ_j form perfect systems for $j = 1, \ldots, r$, with corresponding orthogonal polynomials $\{P_n^{(j)}\}_n$. Then the system of r measures $\mu_1, \ldots, \mu_r$ is perfect if and only if, for every multi-index $\vec{n}$, there exists a polynomial $P_{\vec{n}}$ and scalars $c_{\vec{n},k}^{(j)}$ so that*

$$P_{\vec{n}}(z) = \sum_{\ell=n_j}^{|\vec{n}|} c_{\vec{n},\ell}^{(j)} P_\ell^{(j)}(z), \quad c_{\vec{n},n_j}^{(j)} \neq 0, \quad c_{\vec{n},|\vec{n}|}^{(j)} \neq 0, \quad j = 1, \ldots, r. \tag{3.14}$$

In this case, $P_{\vec{n}}$ is the (*up to normalization unique*) *multiple orthogonal polynomial of type* II *with respect to $\vec{n}$.*

Proof. If $\vec{n}$ is normal and $P_{\vec{n}}$ is the corresponding multiple orthogonal polynomial, then (3.14) follows by taking

$$c_{\vec{n},\ell}^{(j)} = \frac{\displaystyle\int P_{\vec{n}}(z) P_\ell^{(j)}(z)\, \mathrm{d}\mu_j(z)}{\displaystyle\int \left(P_\ell^{(j)}(z)\right)^2 \mathrm{d}\mu_j(z)}, \tag{3.15}$$

where we observe that the denominator is not zero according to (3.13) for the singleton system μ_j. In addition, $c_{\vec{n},\ell}^{(j)} = 0$ for $\ell < n_j$ by (3.12), $c_{\vec{n},n_j}^{(j)} \neq 0$ by (3.13), and $c_{\vec{n},|\vec{n}|}^{(j)} \neq 0$.

Conversely, (3.14) plus the perfectness of the singleton system μ_j implies (3.12),(3.13), and hence the system $\mu_1, \ldots, \mu_r$ is perfect by Lemma 3.4. □

We now prove Theorem 3.3 by showing that (the analytic extension of) the integral expression (3.11) is a possible candidate for a formal multiple Wilson polynomial.

Proof of Theorem 3.3. According to assumptions (3.9) and (3.10) of Theorem 3.3, we find that $\alpha_j, \beta, \alpha_j + \beta + 1 \in \mathbb{C} \setminus \{-1, -2, \ldots\}$, $j = 1, \ldots, r$, and $\alpha_i - \alpha_j \notin \mathbb{Z}$ whenever $i \neq j$. It follows from Theorem 3.1 that the Jacobi–Piñeiro system $\mu^{(\alpha_j,\beta)}$, $j = 1, \ldots, r$, is perfect. From Lemma 3.5 we may conclude that there exist scalars $c_{\vec{n},\ell}^{(j)}$ so that

$$P_{\vec{n}}^{(\vec{\alpha},\beta)}(z) = \sum_{\ell=n_j}^{|\vec{n}|} c_{\vec{n},\ell}^{(j)} P_\ell^{(\alpha_j,\beta)}(z), \quad c_{\vec{n},n_j}^{(j)} \neq 0, \quad c_{\vec{n},|\vec{n}|}^{(j)} \neq 0,$$

$j = 1, \ldots, r$. From, e.g., (3.4) we see that Jacobi–Piñeiro polynomials are rational functions in each of the parameters α_j or β. Taking into account (2.4) and (3.15), we may conclude that any of the coefficients $c_{\vec{n},\ell}^{(j)}$ is a meromorphic function in each of the parameters α_j or β. We now introduce

$$q_{\vec{n}}(z^2; a, \vec{b}, c, d) = \kappa_{\vec{n}} \int_0^1 P_{\vec{n}}^{(\vec{\alpha},\beta)}(u) w^{(a-1,\beta)}(u) K(u, z; a, 0, c, d)\, \mathrm{d}u.$$

This function is well defined for $0 < |\Re(z)| < \Re(a)$ if $\Re(a) > 0$ and $\Re(c+d) > 0$. However, using (2.9), we obtain for every $j = 1, \ldots, r$ that

$$\begin{aligned} q_{\vec{n}}(z^2; a, \vec{b}, c, d) &= \kappa_{\vec{n}} \sum_{\ell=n_j}^{|\vec{n}|} c_{\vec{n},\ell}^{(j)} \int_0^1 P_\ell^{(\alpha_j,\beta)}(u) w^{(\alpha_j,\beta)}(u) K(u, z; a, b_j, c, d)\, \mathrm{d}u, \\ &= \sum_{\ell=n_j}^{|\vec{n}|} \frac{\vec{n}!}{\ell!} \frac{\Gamma(a+c+|\vec{n}|)\Gamma(a+d+|\vec{n}|)}{\Gamma(a+c+\ell)\Gamma(a+d+\ell)} c_{\vec{n},\ell}^{(j)}\, p_\ell(z^2; a, b_j, c, d), \end{aligned}$$

and thus

$$\begin{aligned} q_{\vec{n}}(z^2; a, \vec{b}, c, d) &= \sum_{\ell=n_j}^{|\vec{n}|} \frac{\vec{n}!}{\ell!} (a+c+\ell)_{|\vec{n}|-\ell}\, (a+d+\ell)_{|\vec{n}|-\ell} c_{\vec{n},\ell}^{(j)}\, p_\ell(z^2; a, b_j, c, d) \\ &= \sum_{\ell=n_j}^{|\vec{n}|} d_{\vec{n},\ell}^{(j)}\, p_\ell(z^2; a, b_j, c, d). \end{aligned} \tag{3.16}$$

Observing that the expressions on the right-hand side of (3.16) are polynomials in z and meromorphic in any of the parameters $a, \vec{b}, c, d$, we see that the right-hand expression of (3.16) is well defined and independent of j for any choice of z and the parameters $a, \vec{b}, c, d$, as long as (3.9) and (3.10) hold and $b_i - b_j \notin \mathbb{Z}$, $i \neq j$. Moreover, the new coefficients $d_{\vec{n},n_j}^{(j)}$ and $d_{\vec{n},|\vec{n}|}^{(j)}$ are different from zero.

Thus, $q_{\vec{n}}(\cdot; a, \vec{b}, c, d)$ defined by (3.16) is a suitable candidate for a formal multiple Wilson polynomial, and the system of Wilson measures is perfect by Lemma 3.5. $\Box$

We now want to deduce explicit expressions for the formal multiple Wilson polynomials based on the explicit expressions (3.4) and (3.5) for the Jacobi–Piñeiro polynomials. Here we use the Kampé de Fériet series [24]

$$\begin{aligned} &F_{q:q_1;q_2}^{p:p_1;p_2}\left(\begin{array}{c} \vec{f} : \vec{g}; \vec{h} \\ \vec{\phi} : \vec{\psi}; \vec{\xi} \end{array} \middle| z_1, z_2 \right) \\ &:= \sum_{k=0}^{\infty} \frac{\prod_{\ell=1}^{p} (f_\ell)_k}{\prod_{\ell=1}^{q} (\phi_\ell)_k} \sum_{j=0}^{k} \frac{\prod_{\ell=1}^{p_1} (g_\ell)_{k-j} \prod_{\ell=1}^{p_2} (h_\ell)_j}{\prod_{\ell=1}^{q_1} (\psi_\ell)_{k-j} \prod_{\ell=1}^{q_2} (\xi_\ell)_j} \frac{z_1^{k-j}}{(k-j)!} \frac{z_2^j}{j!}, \end{aligned} \tag{3.17}$$

which are a generalization of the 4 Appell series in 2 variables. Notice that, for $p = q = 0$, the Kampé de Fériet series is a product of two hypergeometric series. Also, in the case $r = 2$, our functions $\mathcal{M}_{q,\vec{n}}^{p;m}$ defined in (3.3) are (finite) Kampé de Fériet series

$$\begin{aligned} &\mathcal{M}_{q,(n_1,n_2)}^{p;m}\left(\begin{array}{c} \vec{f}; g_1 : \cdots : g_m \\ \vec{\phi}; \psi_1 : \cdots : \psi_m \end{array} \middle| (z_1, z_2) \right) \\ &= F_{q:0;r}^{p:1;m+1}\left(\begin{array}{c} \vec{f} : (-n_1); (-n_2, g_1, \ldots, g_m) \\ \vec{\phi} : (); (\psi_1, \ldots, \psi_m) \end{array} \middle| z_1, z_2 \right). \end{aligned} \tag{3.18}$$

In what follows the parameters in the Kampé de Fériet series will always be chosen so that the sum in (3.17) is finite, and hence we are not concerned with convergence problems.

Corollary 3.6. *Let $\vec{e} = (1, \dots, 1)$ be a multi-index of length r, $s(\vec{n}) = (n_1, n_1+n_2, \dots, |\vec{n}|)$ and $\sigma_j = a + b_j + c + d - 1$, $j = 1, \dots, r$. Denote by $\vec{v}^{(j)}$ the vector $\vec{v}$ without the jth component. For the multiple Wilson polynomials we have the hypergeometric representations*

$$\begin{aligned} p_{\vec{n}}(z^2; a, \vec{b}, c, d) &= (a\vec{e} + \vec{b})_{\vec{n}} (a + c)_{|\vec{n}|} (a + d)_{|\vec{n}|} \\ &\times \mathcal{M}^{3;2}_{3,\vec{n}} \left(\begin{matrix} (a - z, a + z, \sigma_1 + n_1); (a\vec{e} + \vec{b} + \vec{n})^{(r)} : (\vec{\sigma} + s(\vec{n}))^{(1)} \\ (a + c, a + d, a + b_1); (a\vec{e} + \vec{b})^{(1)} : (\vec{\sigma} + s(\vec{n}))^{(r)} \end{matrix} \middle| \vec{e} \right) \end{aligned} \tag{3.19}$$

and

$$\begin{aligned} &p_{\vec{n}}(z^2; a, \vec{b}, c, d) \\ &\quad = (a\vec{e} + \vec{b})_{\vec{n}} (a + c)_{|\vec{n}|} (a + d)_{|\vec{n}|} \\ &\qquad \times F^{2:1;r+1}_{2:0;r} \left(\begin{matrix} (a - z, a + z) : (c + d - 1); (a\vec{e} + \vec{b} + \vec{n}, 1 - c - d - |\vec{n}|) \\ (a + c, a + d) : (); (a\vec{e} + \vec{b}) \end{matrix} \middle| 1, 1 \right), \end{aligned} \tag{3.20}$$

where $(a\vec{e} + \vec{b})_{\vec{n}} = \prod_{j=1}^{r} (a + b_j)_{n_j}$.

Proof. First note that $(1 - z)^{-\beta} = \sum_{\ell=0}^{\infty} (\beta)_l \frac{z^\ell}{\ell!}$, which converges in the unit disk, so that expression (3.5) can then be written as

$$P^{(\vec{\alpha},\beta)}_{\vec{n}}(z) = \frac{(\vec{\alpha} + \vec{e})_{\vec{n}}}{\vec{n}!} F^{0:1;r+1}_{0:0;r} \left(\begin{matrix} - : (\beta); (\vec{\alpha} + \vec{n} + \vec{e}, -\beta - |\vec{n}|) \\ - : -; (\vec{\alpha} + \vec{e}) \end{matrix} \middle| z, z \right). \tag{3.21}$$

Then start from the Jacobi–Piñeiro transform (3.11) and replace the Jacobi–Piñeiro polynomial $P^{(\vec{\alpha},\beta)}_{\vec{n}}$ by its explicit expressions (3.4) and (3.21), respectively. Since the sums are finite, we can interchange the integral with the sums. Applying (2.10) then completes the proof. □

Remark 3.7. For $r = 2$, we may apply (3.18) to (3.19), leading to a representation as a Kampé de Fériet series of type $F^{3:1;3}_{3:0;2}$. It seems to be non-trivial to derive from this formula the representation as a Kampé de Fériet series of type $F^{2:1;3}_{2:0;2}$ as in (3.20).

4. Limit relations

In this section we consider some cases in which the orthogonality conditions of the formal multiple Wilson polynomials reduce to orthogonality conditions with respect to a positive measure on the real line. We then recover multiple Wilson and multiple Racah polynomials. Next we use the limit relations between the orthogonal polynomials in the Askey table [17] to obtain some new examples of multiple orthogonal polynomials of type II and some known examples. In particular we look at what happens with the explicit expressions (3.19) and (3.20) after applying these limit relations, where we use the notation $\vec{e} = (1, \dots, 1) \in \mathbb{R}^r$ and $s(\vec{n}) = (n_1, n_1 + n_2, \dots, |\vec{n}|)$. Most of these examples are known to be AT systems, see [5,11], which implies that every multi-index is normal .

4.1. Multiple Wilson

With some restrictions on the parameters the orthogonality conditions of the formal multiple Wilson polynomials reduce to real orthogonality conditions with respect to positive measures on the real line. Let $b_j > 0$, $j = 1, \ldots, r$, $b_i - b_j \notin \mathbb{Z}$ whenever $i \neq j$, $\Re(a), \Re(c), \Re(d) > 0$ and a, c, d be real except for a conjugate pair. In that case the imaginary axis can be taken as the contour $\mathcal{C}$. The multiple Wilson polynomials

$$W_{\vec{n}}(z^2; a, \vec{b}, c, d) := p_{\vec{n}}(-z^2; a, \vec{b}, c, d) \tag{4.1}$$

then satisfy the real orthogonality relations

$$\int_0^\infty (x^2)^m \, W_{\vec{n}}(x^2; a, \vec{b}, c, d) \left| \frac{\Gamma(a+ix)\Gamma(b_j+ix)\Gamma(c+ix)\Gamma(d+ix)}{\Gamma(2ix)} \right|^2 \mathrm{d}x = 0,$$

$0 \leqslant m \leqslant n_j - 1$, $j = 1, \ldots, r$. If $a < 0$, $a + b_j > 0$, $j = 1, \ldots, r$, and $a + c, a + d$ are positive or a pair of complex conjugates with positive real parts, then we obtain the same orthogonality conditions but with some extra positive point masses.

4.2. Multiple Racah

As in the scalar case it is also possible to obtain a purely discrete orthogonality. The multiple Racah polynomials $R_{\vec{n}}(\cdot; \vec{\alpha}, \beta, \gamma, \delta)$, where we only change the parameter α with $\alpha_i - \alpha_j \notin \mathbb{Z}$ whenever $i \neq j$, satisfy the discrete orthogonality

$$\sum_{k=0}^{N} \frac{(\alpha_j+1)_k(\gamma+1)_k(\beta+\delta+1)_k(\gamma+\delta+1)_k((\gamma+\delta+3)/2)_k}{(-\alpha_j+\gamma+\delta+1)_k(-\beta+\gamma+1)_k((\gamma+\delta+1)/2)_k(\delta+1)_k k!} \times R_{\vec{n}}(\lambda(k); \vec{\alpha}, \beta, \gamma, \delta)\,(\lambda(k))^m = 0,$$

$0 \leqslant m \leqslant n - 1$, $j = 1, \ldots, r$, where

$$\lambda(z) = z(z+\gamma+\delta+1) \quad \text{and} \quad \beta+\delta+1 = -N \ \text{ or } \ \gamma+1 = -N.$$

They can be found by applying the substitution $z \to z + a$ and the change of variables $\alpha_j = a + b_j - 1, \beta = c + d - 1, \gamma = a + d - 1, \delta = a - d$ on the polynomials $p_{\vec{n}}(z^2; a, \vec{b}, c, d)/((a\vec{e} + \vec{b})_{\vec{n}}(a + c)_{|\vec{n}|}(a + d)_{|\vec{n}|})$ and we need a translation of conditions (3.9) and (3.10). For the multiple Racah polynomials we then have the expressions

$$R_{\vec{n}}(\lambda(z); \vec{\alpha}, \beta, \gamma, \delta) = \mathcal{M}^{3;2}_{3,\vec{n}} \left(\begin{array}{c} (-z, z+\gamma+\delta+1, \sigma_1+n_1); (\vec{\alpha}+\vec{n}+\vec{e})^{(r)} : (\vec{\sigma}+s(\vec{n}))^{(1)} \\ (\beta+\delta+1, \gamma+1, \alpha_1+1); (\vec{\alpha}+\vec{e})^{(1)} : (\vec{\sigma}+s(\vec{n}))^{(r)} \end{array} \middle| \, \vec{e} \right)$$

with $\sigma_j = \alpha_j + \beta + 1$, $j = 1, \ldots, r$, and

$$R_{\vec{n}}(\lambda(z); \vec{\alpha}, \beta, \gamma, \delta) = F^{2:1;r+1}_{2:0;r} \left(\begin{array}{c} (-z, z+\gamma+\delta+1) : (\beta); (\vec{\alpha}+\vec{n}+\vec{e}, -\beta-|\vec{n}|) \\ (\beta+\delta+1, \gamma+1) : (); (\vec{\alpha}+\vec{e}) \end{array} \middle| \, 1, 1 \right).$$

An example of sufficient conditions to have positive weights is

$$\beta+\delta+1=-N,\quad \gamma+\delta+1>-1,\quad \alpha_j>-1,\quad \gamma+\delta+1>-\alpha_j,$$

$j=1,\dots,r$, and either

$$\gamma+1>-N \quad \text{or} \quad \delta+1>-N.$$

Remark 4.1. Recall that the Wilson weight is symmetric in the four parameters so that we can switch these parameters in the change of variables. We then obtain multiple Racah polynomials where we change other parameters. For example we have multiple Racah polynomials $R_{\vec{n}}(\cdot;\alpha,\vec{\beta},\gamma,\delta)$ where we only change the parameter β in the weights ($\beta_i-\beta_j\notin\mathbb{Z}$ whenever $i\neq j$). In that case we have that $\alpha+1=-N$ or $\gamma+1=-N$. As a second example it is possible to change the parameters β,γ and δ in such a way that $\delta_j+\gamma_j$ and $\delta_j+\beta_j$ do not change and $\gamma_i-\gamma_j\notin\mathbb{Z}$ whenever $i\neq j$. Here we assume that $\alpha+1=-N$ or $\beta_j+\delta_j+1=-N$ for every j. We then denote these multiple Racah polynomials by $R_n(\cdot;\alpha,\vec{\beta},\vec{\gamma},\vec{\delta})$. However, this does not give another family of polynomials because

$$R_{\vec{n}}(\lambda(z);\alpha,\vec{\beta},\gamma,\delta)=R_{\vec{n}}(\lambda(z);\vec{\beta}+\delta\vec{e},\alpha-\delta,\gamma,\delta) \tag{4.2}$$

and

$$R_{\vec{n}}(\lambda(z);\alpha,\vec{\beta},\vec{\gamma},\vec{\delta})=R_{\vec{n}}(\lambda(z);\vec{\gamma},\alpha+\beta_j-\gamma_j,\alpha,\gamma_j+\delta_j-\alpha). \tag{4.3}$$

These relations will help us in some of the examples of the subsections below to find explicit expressions for the polynomials.

4.3. Some new examples

4.3.1. Multiple continuous dual Hahn

Let $b_i-b_j\notin\mathbb{Z}$ whenever $i\neq j$. The multiple continuous dual Hahn polynomials satisfy the orthogonality conditions of the multiple Wilson polynomials where we let $d\to+\infty$ (after dividing by $\Gamma(d)^2$). Similarly we obtain real orthogonality conditions with respect to a positive measure if $b_j>0$ and a,c are positive or a pair of complex conjugates with positive real parts. We then denote the $\vec{n}$th multiple continuous dual Hahn polynomial by $S_{\vec{n}}(\cdot;a,\vec{b},c)$. These polynomials satisfy the orthogonality conditions

$$\int_0^\infty (x^2)^m\, S_{\vec{n}}(x^2;a,\vec{b},c)\left|\frac{\Gamma(a+ix)\Gamma(b_j+ix)\Gamma(c+ix)}{\Gamma(2ix)}\right|^2 \mathrm{d}x=0,$$

$0\leqslant m\leqslant n_j-1$, $j=1,\dots,r$. It is clear that

$$S_{\vec{n}}(z^2;a,\vec{b},c)=\lim_{d\to+\infty}\frac{W_{\vec{n}}(z^2;a,\vec{b},c,d)}{(a+d)_{|\vec{n}|}} \tag{4.4}$$

so, by

$$\lim_{\alpha\to+\infty}\frac{(c+d-1)_{k-j}\,(1-c-d-|\vec{n}|)_j}{(a+d)_k}=(-1)^j,\quad 0\leqslant j\leqslant k,$$

the multiple continuous dual Hahn polynomials have the explicit expressions

$$
\begin{aligned}
&S_{\vec{n}}(z^2; a, \vec{b}, c) \\
&\quad = (a\vec{e}+\vec{b})_{\vec{n}}(a+c)_{|\vec{n}|}\, \mathcal{M}^{2;1}_{2,\vec{n}}\left(\begin{matrix} (a-iz, a+iz); (a\vec{e}+\vec{b}+\vec{n})^{(r)} \\ (a+c, a+b_1); (a\vec{e}+\vec{b})^{(1)} \end{matrix} \,\middle|\, \vec{e} \right), \\
&\quad = (a\vec{e}+\vec{b})_{\vec{n}}(a+c)_{|\vec{n}|}\, F^{2:0;r}_{1:0;r}\left(\begin{matrix} (a-iz, a+iz) : (); (a\vec{e}+\vec{b}+\vec{n}) \\ (a+c) : (); (a\vec{e}+\vec{b}) \end{matrix} \,\middle|\, 1, -1 \right).
\end{aligned}
$$

4.3.2. Multiple dual Hahn

Consider γ_j, δ_j, $j = 1, \ldots, r$, so that $\gamma_j, \delta_j > -1$ or $\gamma_j, \delta_j < -N$ for each j and that $\gamma_j + \delta_j$ is independent of j. Suppose also that $\gamma_i - \gamma_j \notin \mathbb{Z}$ whenever $i \neq j$. The multiple dual Hahn polynomials, denoted by $R_{\vec{n}}(\cdot; \vec{\gamma}, \vec{\delta}, N)$, satisfy the system of discrete orthogonality conditions

$$
\sum_{k=0}^{N} \frac{(2k+\gamma_j+\delta_j+1)(\gamma_j+1)_k(-N)_k N!}{(-1)^k(k+\gamma_j+\delta_j+1)_{N+1}(\delta_j+1)_k k!} R_{\vec{n}}(\lambda(k); \vec{\gamma}, \vec{\delta}, N)\,(\lambda(k))^m = 0,
$$

$0 \leqslant m \leqslant n-1$, $j = 1, \ldots, r$, where $\lambda(z) = z(z+\gamma+\delta+1)$. The multiple dual Hahn polynomials are related to the multiple Racah polynomials by

$$
\begin{aligned}
&R_{\vec{n}}(\lambda(z); \vec{\gamma}, \vec{\delta}, N) \\
&\quad = \lim_{\alpha\to+\infty} R_{\vec{n}}(\lambda(z); \alpha, -\vec{\delta}-(N+1)\vec{e}, \vec{\gamma}, \vec{\delta}) \qquad (4.5)\\
&\quad = \lim_{\alpha\to+\infty} R_{\vec{n}}(\lambda(z); \vec{\gamma}, \alpha-\gamma_j-\delta_j-N-1, \alpha, \gamma_j+\delta_j-\alpha), \qquad (4.6)
\end{aligned}
$$

where we use (4.3). Note that

$$
\lim_{\alpha\to+\infty} \frac{(\alpha-\gamma_j-\delta_j-N-1)_{k-j}\,(-\alpha+\gamma_j+\delta_j+N+1-|\vec{n}|)_j}{(\alpha+1)_k} = (-1)^j,
$$

$0 \leqslant j \leqslant k$, so that the multiple dual Hahn polynomials then have the explicit expressions

$$
\begin{aligned}
R_{\vec{n}}(\lambda(z); \vec{\gamma}, \vec{\delta}, N) &= \mathcal{M}^{2;1}_{2,\vec{n}}\left(\begin{matrix} (-z, z+\gamma_j+\delta_j+1); (\vec{\gamma}+\vec{n}+\vec{e})^{(r)} \\ (-N, \gamma_1+1); (\vec{\gamma}+\vec{e})^{(1)} \end{matrix} \,\middle|\, \vec{e} \right) \\
&= F^{2:0;r}_{1:0;r}\left(\begin{matrix} (-z, z+\gamma_j+\delta_j+1) : (); (\vec{\gamma}+\vec{n}+\vec{e}) \\ (-N) : (); (\vec{\gamma}+\vec{e}) \end{matrix} \,\middle|\, 1, -1 \right).
\end{aligned}
$$

4.3.3. Multiple Meixner–Pollaczek

The multiple Meixner–Pollaczek polynomials $P^{(\lambda)}_{\vec{n}}(\cdot; \vec{\phi})$ are multiple orthogonal polynomials (of type II) associated with the system of weights $e^{(2\phi_j-\pi)x}|\Gamma(\lambda+ix)|^2$ on the positive real axis, where $\lambda > 0$, $0 < \phi_j < \pi$, $j = 1, \ldots, r$, and the $\phi_1, \ldots, \phi_r$ are different. These weights form an AT system, see [22, p.141]. The multiple Meixner–Pollaczek polynomials satisfy the conditions

$$\int_0^\infty x^m\ P_{\vec{n}}^{(\lambda)}(x;\vec{\phi})e^{(2\phi_j-\pi)x}|\Gamma(\lambda+ix)|^2\mathrm{d}x=0,\quad 0\leqslant m\leqslant n_j-1,$$

$j=1,\dots,r$. Similar as [17, (2.3.1)] it is easy to check that

$$P_{\vec{n}}^{(\lambda)}(z;\vec{\phi})=\lim_{t\to+\infty}\frac{S_{\vec{n}}((z-t)^2;\lambda+it,t\cot\vec{\phi},\lambda-it)}{(t\csc\vec{\phi})_{\vec{n}}\ \vec{n}!},\tag{4.7}$$

where $t\ \cot\ \vec{\phi}=(t\ \cot\ \phi_1,\dots,t\cot\phi_r)$ and $t\csc\vec{\phi}=(t\csc\phi_1,\dots,t\csc\phi_r)$. The multiple Meixner–Pollaczek polynomials then have the explicit expression

$$P_{\vec{n}}^{(\lambda)}(z;\vec{\phi})=\frac{(2\lambda)_{|\vec{n}|}\prod_{j=1}^r e^{in_j\phi_j}}{\vec{n}!}\ \mathcal{M}_{1,\vec{n}}^{1;0}\left(\begin{matrix}(\lambda+iz);-\\(2\lambda);-\end{matrix}\middle|\ \vec{e}-e^{-2i\vec{\phi}}\right),$$

where $e^{-2i\vec{\phi}}=(e^{-2i\phi_1},\dots,e^{-2i\phi_r})$. Here we do not have a Kampé de Fériét representation such as in (3.20).

4.3.4. *Formal multiple continuous Hahn*

Similar as in [17, (2.1.2)] we can use the limit relation

$$P_{\vec{n}}(z;a,\vec{b},c,d)=\lim_{t\to\infty}\frac{p_{\vec{n}}((z+t)^2;a-t,\vec{b}+t\vec{e},c-t,d+t)}{(a+c-2t)_{|\vec{n}|}\ \vec{n}!}\tag{4.8}$$

in order to find the formal continuous Hahn polynomials. They have the explicit expressions

$$\begin{aligned}&\frac{P_{\vec{n}}(z;a,\vec{b},c,d)}{(a\vec{e}+\vec{b})_{\vec{n}}(a+d)_{|\vec{n}|}}\\&\quad=\mathcal{M}_{2,\vec{n}}^{2;2}\left(\begin{matrix}(a+z,\sigma_1+n_1);(a\vec{e}+\vec{b}+\vec{n})^{(r)}:(\vec{\sigma}+s(\vec{n}))^{(1)}\\(a+d,a+b_1);(a\vec{e}+\vec{b})^{(1)}:(\vec{\sigma}+s(\vec{n}))^{(r)}\end{matrix}\middle|\ \vec{e}\right)\\&\quad=F_{1:0;r}^{1:1;r+1}\left(\begin{matrix}(a+z):(c+d-1);(a\vec{e}+\vec{b}+\vec{n},1-c-d-|\vec{n}|)\\(a+d):();(a\vec{e}+\vec{b})\end{matrix}\middle|\ 1,1\right),\end{aligned}$$

where $\sigma_j=a+b_j+c+d-1$, $j=1,\dots,r$. If the parameters satisfy (3.9) and (3.10) and $b_i-b_j\notin\mathbb{Z}$ whenever $i\neq j$, then these polynomials satisfy the orthogonality conditions

$$\int_{\mathcal{C}}P_{\vec{n}}(z;a,\vec{b},c,d)\Gamma(a+z)\Gamma(b_j-z)\Gamma(c+z)\Gamma(d-z)z^m\,\mathrm{d}z=0,\tag{4.9}$$

$0\leqslant m\leqslant n_j-1$, $j=1,\dots,r$, where $\mathcal{C}$ is a contour which is the imaginary axis deformed so as to separate the increasing sequences of poles $(\{b_1+k\}_{k=0}^\infty,\dots,\{b_r+k\}_{k=0}^\infty,\{d+k\}_{k=0}^\infty)$ from the decreasing ones $(\{-a-k\}_{k=0}^\infty,\{-c-k\}_{k=0}^\infty)$.

Remark 4.2. In the scalar case ($r=1$) it is possible to obtain real orthogonality relations with respect to a positive measure if we suppose $\Re(a),\Re(b),\Re(c),\Re(d)>0$ and $a=\bar{b}$, $c=\bar{d}$. This is not possible in this multiple case. For that one needs another family of multiple continuous Hahn polynomials in which one changes both the parameters a and b.

4.4. Some classical discrete multiple orthogonal polynomials

In this section we obtain hypergeometric formulas for the classical discrete examples of multiple orthogonal polynomials of type II, introduced in [5], which are all examples of AT systems. In particular we use the limit relations between these polynomials and the Racah polynomials [17]. Their $\mathcal{M}^{p;r}_{q,\vec{n}}$ representation is already known in the cases $r = 1, 2$. Where it exists, the explicit expression in terms of a Kampé de Fériét series is new. We denote by δ_k the Dirac measure at the point k.

4.4.1. Multiple Hahn

These multiple orthogonal polynomials (of type II) satisfy orthogonality conditions with respect to m hypergeometric distributions

$$\mu_j = \sum_{k=0}^{N} \frac{(\alpha_j+1)_k}{k!}\frac{(\beta+1)_{N-k}}{(N-k)!}\delta_k, \quad \alpha_j > -1,\ \beta > -1,$$

$\alpha_i - \alpha_j \notin \{0, 1, \dots, N-1\}$, $i \neq j$, on the integers $0, \dots, N$. They can be found from the multiple Racah polynomials taking $\gamma + 1 = -N$ and $\delta \to +\infty$, so that

$$\begin{aligned} Q_{\vec{n}}^{\vec{\alpha};\beta;N}(z) &= \mathcal{M}^{2;2}_{2,\vec{n}}\left(\left. \begin{matrix} (-z, \sigma_1+n_1); (\vec{\alpha}+\vec{n}+\vec{e})^{(r)} : (\vec{\sigma}+s(\vec{n}))^{(1)} \\ (-N, \alpha_1+1); (\vec{\alpha}+\vec{e})^{(1)} : (\vec{\sigma}+s(\vec{n}))^{(r)} \end{matrix} \right| \vec{e} \right) \\ &= F^{1:1;r+1}_{1:0;r}\left(\left. \begin{matrix} (-z) : (\beta); (\vec{\alpha}+\vec{n}+\vec{e}, -\beta-|\vec{n}|) \\ (-N) : (); (\vec{\alpha}+\vec{e}) \end{matrix} \right| 1, 1 \right), \end{aligned}$$

where $\sigma_j = \alpha_j + \beta + 1$, $j = 1, \dots, r$. Changing only the parameter β does not give another family of polynomials because of $Q_{\vec{n}}^{\alpha;\vec{\beta};N}(x) = C\, Q_{\vec{n}}^{\vec{\beta};\alpha;N}(N-x)$ with C some constant (depending on $\vec{n}$, α and $\vec{\beta}$). However, we will need an explicit formula in powers of x for these polynomials to obtain multiple Meixner I and multiple Laguerre II. Using (4.2) and the limits we mentioned above, we find that

$$Q_{\vec{n}}^{\alpha;\vec{\beta};N}(z) = \mathcal{M}^{2;2}_{2,\vec{n}}\left(\left. \begin{matrix} (-z, \alpha+\beta_1+n_1+1); (\vec{\beta}+s(\vec{n})+(\alpha+1)\vec{e})^{(1)} \\ (-N, \alpha+1); (\vec{\beta}+s(\vec{n})+(\alpha+1)\vec{e})^{(r)} \end{matrix} \right| \vec{e} \right).$$

4.4.2. Multiple Meixner I

In this case we consider r negative binomial distributions

$$\mu_j = \sum_{k=0}^{\infty} \frac{(\beta)_k\, c_j^k}{k!}\delta_k, \quad 0 < c_j < 1,\ \beta > 0,$$

with all the c_j, $j = 1\dots, r$, different. We get these polynomials from the multiple Hahn polynomials $Q_{\vec{n}}^{\alpha;\vec{\beta};N}$ replacing $\alpha = \beta - 1$, $\beta_j = N\frac{1-c_j}{c_j}$ and letting $N \to +\infty$. We then obtain

$$M_{\vec{n}}^{\beta;\vec{c}}(z) = \mathcal{M}_{1,\vec{n}}^{1;0}\left(\begin{matrix} (-z);\,() \\ (\beta);\,() \end{matrix} \,\middle|\, \frac{\vec{c}-\vec{e}}{\vec{c}} \right),$$

where $\frac{\vec{c}-\vec{e}}{\vec{c}} = \left(\frac{c_1-1}{c_1}, \ldots, \frac{c_r-1}{c_r}\right)$.

4.4.3. Multiple Meixner II

In the case of multiple Meixner II polynomials we only change the parameter β in the negative binomial distributions, so that

$$\mu_j = \sum_{k=0}^{\infty} \frac{(\beta_j)_k\, c^k}{k!}\delta_k, \quad 0 < c < 1,\ \beta_j > 0,$$

with $\beta_i - \beta_j \notin \mathbb{Z}$ whenever $i \neq j$. Taking $\alpha_j = \beta_j - 1$, $\beta = N\frac{1-c}{c}$ and letting $N \to +\infty$ in the explicit formulas for the multiple Hahn polynomials $Q_{\vec{n}}^{\vec{\alpha};\beta;N}$, we obtain

$$\begin{aligned} M_{\vec{n}}^{\vec{\beta};c}(z) &= \mathcal{M}_{1,\vec{n}}^{1;1}\left(\begin{matrix} (-z);\,(\vec{\beta}+\vec{n})^{(r)} \\ (\beta_1);\,(\vec{\beta})^{(1)} \end{matrix} \,\middle|\, \frac{c-1}{c}\,\vec{e} \right) \\ &= F_{0:0;r}^{1:0;r}\left(\begin{matrix} (-z) : ();\,(\vec{\beta}+\vec{n}) \\ () : ();\,(\vec{\beta}) \end{matrix} \,\middle|\, \frac{c-1}{c}, \frac{1-c}{c} \right). \end{aligned}$$

4.4.4. Multiple Kravchuk

These polynomials satisfy the orthogonality conditions (1.1) with the r binomial distributions

$$\mu_j = \sum_{k=0}^{N} \binom{N}{k} p_j^k (1-p_j)^{N-k}\delta_k, \quad 0 < p_j < 1,$$

where all the p_j, $j = 1\ldots, r$, are different. They are related to the multiple Hahn polynomials $Q_{\vec{n}}^{\vec{\alpha};\beta;N}$ replacing $\beta = t$, $\alpha_j \to \frac{p_j}{1-p_j}t$ and letting $t \to +\infty$. We then get

$$K_{\vec{n}}^{\vec{p};N}(z) = \mathcal{M}_{1,\vec{n}}^{1;0}\left(\begin{matrix} (-z);\,() \\ (-N);\,() \end{matrix} \,\middle|\, \frac{1}{\vec{p}} \right),$$

where $\frac{1}{\vec{p}} = \left(\frac{1}{p_1}, \ldots, \frac{1}{p_r}\right)$.

4.4.5. Multiple Charlier

In the case of multiple Charlier we consider r Poisson distributions

$$\mu_j = \sum_{k=0}^{\infty} \frac{a_j^k}{k!}\delta_k, \quad a_j > 0,$$

with all the a_j, $j = 1\ldots, r$, different. The corresponding multiple orthogonal polynomials (of type II) can be found from the multiple Meixner I polynomials taking $c_j = \frac{a_j}{a_j+\beta}$ and letting $\beta \to +\infty$. The multiple Charlier polynomials then have the explicit expression

$$C_{\vec{n}}^{\vec{a}}(z) = \mathcal{M}_{0,\vec{n}}^{1;0}\left(\begin{matrix}(-z);\,()\\ ();\,()\end{matrix}\,\middle|\, -\frac{1}{\vec{a}}\right),$$

where $\frac{1}{\vec{a}} = \left(\frac{1}{a_1}, \ldots, \frac{1}{a_r}\right)$.

4.5. Some classical continuous multiple orthogonal polynomials

In this section we recall some classical continuous examples of multiple orthogonal polynomials of type II where the measures (or weight functions) form an AT system and obtain hypergeometric formulas for these polynomials. Their $\mathcal{M}_{q,\vec{n}}^{p;r}$ representation is already known in the cases $r = 1, 2$. The explicit expression in terms of a Kampé de Fériét series is new (if it exists). For an overview of these polynomials and their properties we recommend [4,11].

4.5.1. Jacobi–Piñeiro

In Section 3.1 we recalled the Jacobi–Piñeiro polynomials $P_{\vec{n}}^{\vec{\alpha},\beta}$, which, in the case $\alpha_j, \beta > -1$, are the multiple orthogonal polynomials (of type II) with respect to the Jacobi weights $w^{\alpha_j,\beta}(x) = x^{\alpha_j}(1-x)^{\beta}$, $j = 1, \ldots, r$, on the interval $[0, 1]$. Here $\alpha_i - \alpha_j \notin \mathbb{Z}$ whenever $i \neq j$. For the explicit formulas see Theorem 3.2. Similar as in the multiple Hahn case we have that $P_{\vec{n}}^{(\alpha,\vec{\beta})}(z) = (-1)^{|\vec{n}|}P_{\vec{n}}^{(\vec{\beta},\alpha)}(1-z)$. So, changing only the parameter β does not give another family of polynomials. For these polynomials we have

$$\begin{aligned}
&P_{\vec{n}}^{(\alpha,\vec{\beta})}(z)\\
&\quad= \lim_{N\to+\infty} \frac{(\alpha+1)_{|\vec{n}|}}{\vec{n}!}\, Q_{\vec{n}}^{\alpha;\vec{\beta}:N}(Nz)\\
&\quad= \frac{(\alpha+1)_{|\vec{n}|}}{\vec{n}!}\,\mathcal{M}_{1,\vec{n}}^{1;1}\left(\begin{matrix}(\alpha+\beta_1+n_1+1);\,(\vec{\beta}+s(\vec{n})+(\alpha+1)\vec{e})^{(1)}\\ (\alpha+1);\,(\vec{\beta}+s(\vec{n})+(\alpha+1)\vec{e})^{(r)}\end{matrix}\,\middle|\, z\vec{e}\right).
\end{aligned}$$

4.5.2. Multiple Laguerre I

The multiple Laguerre I polynomials $L_{\vec{n}}^{\vec{\alpha}}$ are orthogonal on $[0, +\infty)$ with respect to the r weights $w_j(x) = x^{\alpha_j}e^{-x}$, where $\alpha_j > -1$, $j = 1, \ldots, r$, and $\alpha_i - \alpha_j \notin \mathbb{Z}$ whenever $i \neq j$. They can be found from the Jacobi–Piñeiro polynomials $P_{\vec{n}}^{\vec{\alpha},\beta}$ substituting $z \to \frac{z}{\beta}$ and letting $\beta \to \infty$. We then obtain the hypergeometric expressions

$$\begin{aligned}
L_{\vec{n}}^{\vec{\alpha}}(z) &= \frac{(\vec{\alpha}+\vec{e})_{\vec{n}}}{\vec{n}!}\,\mathcal{M}_{1,\vec{n}}^{0;1}\left(\begin{matrix}();\,(\vec{\alpha}+\vec{n}+\vec{e})^{(r)}\\ (\alpha_1+1);\,(\vec{\alpha}+\vec{e})^{(1)}\end{matrix}\,\middle|\, z\vec{e}\right)\\
&= \frac{(\vec{\alpha}+\vec{e})_{\vec{n}}}{\vec{n}!}\, e^{z}\,{}_rF_r\left(\begin{matrix}\vec{\alpha}+\vec{n}+\vec{e}\\ \vec{\alpha}+\vec{e}\end{matrix}\,\middle|\, -z\right).
\end{aligned}$$

4.5.3. Multiple Laguerre II

In this case the polynomials $L_{\vec{n}}^{(\alpha,\vec{c})}$ have the orthogonality conditions (1.1) with respect to the weight functions $w_j(x) = x^{\alpha} e^{-c_j x}$, $j = 1, \ldots, r$, on $[0, +\infty)$, where $\alpha > -1$, $c_j > 0$ and all the c_j different. They can be obtained from the Jacobi–Piñeiro polynomials $P_{\vec{n}}^{(\alpha,\vec{\beta})}$ by the substitutions $z \to \frac{z}{t}$, taking $\beta_j = c_j t$ and letting $t \to \infty$. We then get

$$L_{\vec{n}}^{(\alpha,\vec{c})}(z) = \frac{(\alpha+1)_{|\vec{n}|}}{\vec{n}!} \, \mathcal{M}_{1,\vec{n}}^{0;0} \left(\begin{array}{c} ();\,() \\ (\alpha+1);\,() \end{array} \middle| z\vec{c} \right).$$

4.5.4. Multiple Hermite

In the multiple Hermite case we consider the type II multiple orthogonal polynomials $H_{\vec{n}}^{\vec{c}}$ with respect to the weights $w_j(x) = e^{-x^2 + c_j x}$, $j = 1, \ldots, r$, on $(-\infty, +\infty)$. Here the c_j are different real numbers. These polynomials can be obtained from the Jacobi–Piñeiro polynomials $P_{\vec{n}}^{\vec{\alpha},\beta}$ taking $\alpha_j = \beta + c_j\sqrt{\beta}$, the substitution $z \to (z+\sqrt{\beta})/(2\sqrt{\beta})$ and letting $\beta \to +\infty$ after multiplying with some constant depending on $\vec{n}$ and β.

5. Conclusion

In Section 4 we have shown that, for a particular restricted choice of parameters, formal multiple Wilson polynomials contain both multiple Wilson and multiple Racah polynomials. These polynomials can be found on the top of the scheme presented in Fig. 2, which resembles the well-known Askey scheme for classical orthogonal polynomials. Every entry of this scheme corresponds to an extension of classical orthogonal polynomials to the multiple orthogonality case, with measures having the same support. The arrows in this scheme correspond to possible limit relations: most of them have explicitly been given in Sections 4.3–4.5. It is well known that multiple Hahn polynomials and all multiple orthogonal

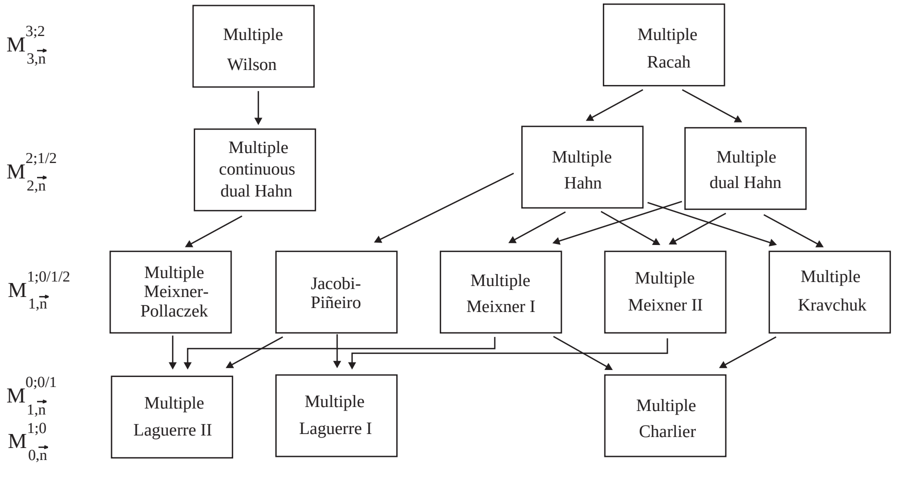

Fig. 2. An (incomplete) multiple AT-Askey scheme.

polynomials occurring in the third and the fourth row of the scheme are examples of AT systems. We conjecture that also the remaining families of measures in the first and the second row of the scheme form an AT-system. This motivates us to call the scheme of Fig. 2 the multiple AT-Askey scheme.

This scheme does not contain all the possible examples of multiple orthogonal polynomials generalizing the classical orthogonal polynomials of the Askey scheme. In [11] the authors also mentioned some examples of Angelesco systems (with their hypergeometric expression). It is also possible to change more than one parameter in the Wilson weight (maybe with some correlation) in order to find other examples of multiple Wilson polynomials. Then it is for example possible to obtain (other kinds of) multiple continuous Hahn polynomials corresponding to positive measures on the real line, using some limit relations.

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